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Volume XV, No. 1 July, 1943

OFFICE OF THE REGISTRAR OF ...
22 SEP 1943
MADRAS
317

JOURNAL

OF THE

MADRAS UNIVERSITY

Section B. Science

PUBLISHED BY THE UNIVERSITY
MADRAS

1943

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Volume XV, No. 1 July, 1943

OFFICE OF THE REGISTRAR OF BOOKS
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ON RAMANUJAN'S TRIGONOMETRICAL SUM $C_m(n)$.

BY

K. G. RAMANATHAN,
University of Madras.

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1. Ramanujan's trigonometrical sum[3] $C_m(n)$ is defined as

$$C_m(n)^* = \sum_{\lambda} e^{\frac{2\pi i n\lambda}{m}} \qquad .. \quad (1)$$

where λ ranges through all the integers less than and prime to m. Bachmann[4] defines R.D. Von Sterneck's function $f(n, m)$† as the excess of the number of partitions of n into an even number of parts (mod m) over those into an odd number of parts; the parts being all distinct and zero being not counted as a part. The object of this paper is to show the identity of these two functions. This fact does not seem to have been noticed before in mathematical literature. I prove also some results given by Ramanujan and others given by Bachmann. The paper contains also a curious result viz.

$$\prod_{\lambda} \sin \frac{\pi\lambda}{m} = 2^{-\varphi(m)} e^{\Lambda(m)} \qquad .. \quad (2)$$

where $\varphi(m)$ is Euler's function and $\Lambda(m)$ is the arithmetical function defined by

$$-\frac{\zeta^1(s)}{\zeta(s)} = \sum_{n=1}^{\infty} \frac{\Lambda(n)}{n^s} \qquad .. \quad (3)$$

*We shall hereafter use Σ and Π to mean the sum and product respectively over all integers less than and prime to m.

†For instance $f(4, 6) = -1$.

$$4 \equiv 1+3 \equiv 4 \equiv 2+3+5 \equiv 1+4+5 \equiv 1+2+3+4 \pmod 6.$$

$\zeta(s)$ being Riemann Zeta function.

I am deeply indebted to Prof. R. Vaidyanathaswami for his helpful suggestions.

Theorem 1.
$$\sum_{k=1}^{m} C_m(k)\, e^{\frac{2n\pi k i}{m}} = o \quad \text{if } (n, m) > 1$$
$$= m \text{ if } (n, m) = 1 \quad .. \quad (4)$$

Let us write $e(x)$ to denote $e^{2\pi i x}$

Now $C_m(k) = \sum_{\lambda} e\left(\frac{k\lambda}{m}\right)$ and therefore

$$\sum_{k=1}^{m} C_m(k)\, e\left(\frac{nk}{m}\right) = \sum_{k=1}^{m} \sum_{\lambda} e\left[\frac{k(\lambda+n)}{m}\right]$$

$$= \sum_{\lambda} \sum_{k=1}^{m} e\left[\frac{k(\lambda+n)}{m}\right]$$

But $\sum_{k=1}^{m} e\left[\frac{k(\lambda+n)}{m}\right]$ vanishes for all λ and n unless n is prime to m when, for $\lambda = m - n$, it has the value m.

Corollary:—(1) $\sum_{k=1}^{m} C_m(k) = 0$ .. (5)

(2) $\sum_{k=1}^{m} C_m(k)\, e\left(\frac{k}{m}\right) = m$ .. (6)

Theorem 2. $C_m(n) = \sum (-1)^{\nu}$ .. (7)

Where ν is the number of partitions of n into distinct parts mod m; Zero being not a part.

It is well-known that

$$\frac{\sin m\theta}{\sin \theta} = 2^{m-1} \sin(\theta+\beta)\sin(\theta+2\beta)\ldots.\sin(\theta+m-1\beta) \quad .. \quad (8)$$

where $\beta = \frac{\pi}{m}$. Making θ tend to zero we have

$$\sin\beta, \sin 2\beta\ldots.\sin\ (m-1)\ \beta = m \mid 2^{m-1} \quad .. \quad (9)$$

Substituting exponential values for the sines and simplifying we have

$$\left[1-e\left(\frac{1}{m}\right)\right]\left[1-e\left(\frac{2}{m}\right)\right]\left[\ldots\ldots\right]\left[1-e\left(\frac{m-1}{m}\right)\right] = m \quad .. \quad (10)$$

using theorem I we have the result (7).

This proves a very curious property of Ramanujan's trigonometrical sum that it is the excess of the number of partitions of n into an even number of parts mod m over these into an odd number of parts. The parts being all distinct and zero being not a part.

Theorem 3. $$C_m(n) = \Sigma\mu\left(\frac{m}{\delta}\right)\delta \quad .. \quad (11)$$

Where δ runs through all the common divisors of m and n and $\mu(n)$ is the arithmetical function[6] defined by

$\mu(n) = o$ if n has a squared factor
$= (-1)^{\nu}$ when n is the product of ν distinct prime factors.

Ramanujan[3] gives a very ingenious proof of this result. I merely reconstruct his proof.

Let $t_1 (= 1), t_2\ t_3 \ldots t_\lambda$ ($\lambda = d(m)$ the number of divisors of m) be the distinct divisors of m.

Then Dr. R. Vaidyanathaswami[1] has shown that the integers mod m can be divided into λ classes $C_1, C_2 \ldots C_\lambda$ in such a way that the numbers in the class C_r are all the integers mod m which have with m a greatest common divisor (g.c.d) equal to t_r. There are thus $\varphi\left(\frac{m}{t_r}\right)$ numbers in the class C_r and the fact that these classes exhaust all the integers mod m is given by Gauss' result

$$\Sigma\, \varphi\left(\frac{m}{t_r}\right) = \Sigma\, \varphi(t_r) = m \qquad \ldots (12)$$

If $a_1, a_2, \ldots\ldots a_q$ $q = \varphi\left(\frac{m}{t}\right)$ be those integers mod m which have with m a g.c.d. equal to t_r then

$$\sum_{p=1}^{q} e\left(\frac{na_p}{m}\right) = \Sigma\, e\left(n\frac{a_p}{t_r} \mid \frac{m}{t_r}\right)$$

$$= C_{\left(\frac{m}{t_r}\right)}(n)$$

so that $$\sum_{k=1}^{m} e\left(\frac{kn}{m}\right) = \sum_{\delta \mid m} C_\delta(n) \qquad \ldots (13)$$

But $\sum_{k=1}^{m} e\left(\frac{kn}{m}\right) = \eta_m(n)$ which[3] vanishes if m does not divide n; and when it divides n it is equal to m. By Moebius' inversion formula[6] we have the Theorem 3.

Theorem 4. If n and n^1 are coprime integers then

$$C_m(n) \cdot C_m(n^1) = \mu(m) \cdot C_m(nn^1) \qquad \text{.. (14)}$$

Using theorem 3 we have

$$C_m(n) \cdot C_m(n^1) = \sum_{\delta | (m, n)} \mu\left(\frac{m}{\delta}\right)\delta \times \sum_{\delta^1 | (m^1, n^1)} \mu\left(\frac{m}{\delta^1}\right)\delta^1$$

$$= \Sigma\, \mu\left(\frac{m}{\delta}\right) \mu\left(\frac{m}{\delta^1}\right) \delta\delta^1$$

But I[5] have proved that if $f(n)$ is any multiplicative function then

$$f(m) \cdot f(n) = f(d) \cdot f\left(\frac{mn}{d}\right) \quad d = (m, n) \qquad \text{.. (15)}$$

$$\therefore \quad C_m(n) \cdot C_m(n^1) = \mu(m)\, \Sigma\, \mu\left(\frac{m}{\delta\delta^1}\right) \delta\delta^1$$

which proves the result (14).

Theorem 5.

$$C_m(n) = \mu\left(\frac{m}{d}\right) \frac{\varphi(m)}{\varphi\left(\frac{m}{d}\right)} \qquad \text{.. (16)}$$

where $d = (m, n)$.

This result is given by Bachmann.

I[5] have proved that if $\varphi(n)$ is Eulers' function

Then

$$\Sigma \frac{\varphi(Mn)}{n^s} = \frac{\varphi(M)M^s}{\varphi_s(M)} \frac{\zeta(s-1)}{\zeta(s)} \qquad .. \quad (17)$$

$$\Sigma \frac{\mu(Mn)}{n^s} = \frac{\mu(M)M^s}{\varphi_s(M)} \frac{1}{\zeta(s)} \qquad .. \quad (18)$$

where $\zeta(s)$ is Riemann Zeta function, and

$$\varphi_s(M) = M^s(1-p_1^{-s})(1-p_2^{-s})\ldots\ldots (M = p_1^{\alpha}p_2^{\beta}\ldots.)$$

From (17) and (18) we easily have

$$\zeta(s-1)\ \Sigma \frac{\mu(Mn)}{n^s} = \frac{\mu(M)}{\varphi(M)} \Sigma \frac{\varphi(Mn)}{n^s}.$$

and by Composition[2] of Dirichlet's series we have

$$\sum_{\delta\delta 1 = n} \mu(M\delta)\delta^1 = \mu(M) \frac{\varphi(Mn)}{\varphi(M)}$$

taking $M = \frac{m}{d}$ and $n = d$ we have the result (16).

3. I shall now prove Ramanujan's[3] interesting theorem viz.,

$$C_m(1) + \frac{1}{2}C_m(2) + \frac{1}{3}C_m(3) + \ldots\ldots = -\Lambda(m) \qquad .. \quad (19)$$

by the use of a certain function $G(m)$.

Theorem 6.

If (*i*) $\zeta(s)$ *is Riemann zeta function*
(*ii*) $\Lambda(m)$ *is defined as in* (3).

and (*iii*) $\prod_{\lambda} \sin \frac{\pi\lambda}{m} = G(m)$

then $\log \; 2^{\varphi(m)} G(m) = \Lambda(m)$ .. (20)

It is easily seen from the property of the integers mod m proved by Dr. R. Vaidyanathaswami and (8) that

$$\Pi^1 \; G\left(\frac{m}{\delta}\right) = m \mid 2^{m-1} \quad .. \quad (21)$$

where the dash denotes that the product is over all divisors of m except unity.

We shall prove some lemmas.

Lemma a. *If* p *is a prime number then*

$$G(p^n) = p \cdot 2^{-\varphi(p^n)} \quad .. \quad (22)$$

$\varphi(n)$ being Euler's function.

This may easily be proved by induction.

For $\prod_{\delta \mid p^{n+1}} G\left(\frac{p^{n+1}}{\delta}\right) = \prod_{d \mid p^n} G\left(\frac{p^n}{d}\right) \times G(p^{n+1})$

$$G(p^{n+1}) = \frac{p^{n+1}}{2^{p^{n+1}} - 1} \times \frac{2^{p} - 1}{p^n} = p \cdot 2^{-\varphi(p^{n+1})}$$

and by the usual argument (22) follows.

Lemma β If p and q are distinct primes then

$$G(pq) = 2^{-\varphi(pq)} \quad \ldots \quad (23)$$

This is easy.

Lemma γ

$$G(p^n q^m) = 2^{-\varphi(p^n q^m)} \quad \ldots \quad (24)$$

Where n and m are positive integers.

This follows from lemma β and using (21). The proof is thus similar to lemma α.

From these lemmas we see that $\log 2^{\varphi(m)} G(m)$ vanishes if m is the product of primes and equals $\log p$ when $m = p^k$ (p being a prime). Thus theorem (6) follows since $\Lambda(m)$ also behaves in the same way.

Corollary:

$$\Pi \sin \frac{\pi \lambda}{m} = 2^{-\varphi(m)} e^{\Lambda(m)} \quad \ldots \quad (25)$$

Now using exponential values for the sines, taking logarithms and simplifying we have

$$\log \prod_{\lambda} \left(1 - e \left(\frac{\lambda}{m} \right) \right) = \log e^{\Lambda(m)} = \Lambda(m) \quad \ldots \quad (26)$$

expanding, rearranging and using (1) we have Ramanujan's curious relation stated in (19).

4. Let us put $m = 6$ in Ramanujan's result (19). We see that $C_6(n) = 2 \cos \frac{n\pi}{3}$ and Ramanujan's theorem reduces to the well-known theorem

$$\operatorname{Cos} \frac{\pi}{3} + \frac{1}{2} \operatorname{Cos} \frac{2\pi}{3} + \frac{1}{3} \operatorname{Cos} \frac{3\pi}{3} + \ldots\ldots = 0. \quad \ldots \quad (27)$$

This can also be seen to follow from very elementary considerations. For, expanding log $(1-z)$ in powers of z and writing $z=ke^{\frac{i\pi}{3}}$ we have the result. Substituting the values of Cos $\frac{\pi}{3}$ etc., we have the interesting theorem.

$$1-\frac{1}{2}-\frac{2}{3}-\frac{1}{4}+\frac{1}{5}+\frac{2}{6}+\frac{1}{7}-\ldots\ldots=0 \qquad \text{(28)}$$

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ON THE CANONICAL EXPRESSION FOR A MEROMORPHIC FUNCTION OF FINITE ORDER

BY

K. CHANDRASEKHARAN.

1. In a paper published in 1941,* I gave a new proof of Hadamard's theorem for the factorization of an integral function of finite order. The method of proof adopted by me was on the lines of Cauchy's proof of the infinite product for sin z. By an exactly similar method, I propose to obtain here a simple and *direct* proof of Nevanlinna's† extension of Hadamard's theorem to the case of meromorphic functions of finite order.

2. Let f(z) be a meromorphic function of finite order. Let (a_n) and (b_n) be the zeros and poles respectively of $f(z)$, arranged in increasing order of moduli, and each counted according to its degree of multiplicity. Let $f(o) \neq o, \infty$. With the notation of Nevanlinna, (which is freely used in this note without further explanation) let $T(r)$ be the characteristic function. Let q be an integer such that

$$\lim_{r \to \infty} \frac{T(r)}{r^{q+1}} = o$$

Then Nevanlinna's theorem states, that

$$f(x) = x^{\alpha} \cdot e^{\sum\limits_{o}^{q} p_{\nu} x^{\nu}} \times$$

$$\lim_{R=\infty} \frac{\prod\limits_{|a_{\nu}|<R} \left(1 - \frac{x}{a_{\nu}}\right) e^{\frac{x}{a_{\nu}} + \cdots\cdots + \frac{1}{q}\left(\frac{x}{a_{\nu}}\right)^{q}}}{\prod\limits_{|b_{\nu}|<R} \left(1 - \frac{x}{b_{\nu}}\right) e^{\frac{x}{b_{\nu}} + \cdots\cdots + \frac{1}{q}\left(\frac{x}{b_{\nu}}\right)^{q}}}$$

uniformly in every circle $|x| \leq \varrho < R$, where α is an integer.

* K. Chandrasekharan: On Hadamard's Factorization Theorem. Jour. Ind. Math. Soc. 5 (1941) pp. 128-132.

† R. Nevanlinna: Le Theoreme de Picard-Borel et la theorie des functions meromorphes. p. 37.

In order to prove the theorem, I require the following

Lemma

$$\left| \frac{f^1(z)}{f(z)} - \sum_{\nu=1}^{M} \frac{1}{z-a_\nu} + \sum_{\nu=1}^{N} \frac{1}{z-b_\nu} \right| < K[T(2R)+O(1)] \frac{R+\varrho}{(R-\varrho)^2}$$

for $|z|=\varrho$, where M and N denote the number of a's and b's within $|z|=R$, and K is a constant, $O<\varrho<R$.

Proof. Let $z=Re^{\theta}$ and $x=\varrho e^{\varphi}$, $\varrho<R$, and

$$g_R(z)=\frac{f(z)}{f(o)} \prod_{\nu=1}^{M} \left(1-\frac{z}{a_\nu}\right)^{-1} \prod_{\nu=1}^{N} \left(1-\frac{z}{b_\nu}\right)$$

Set $\log g_R(z)=h_R(z)$

Then, it is easily seen that*

$$h_R(x)=\frac{1}{2\pi} \int_o^{2\pi} \log|g_R(z)| \frac{z+x}{z-x} d\theta$$

$$=\frac{x}{2\pi} \int_o^{2\pi} \log|g_R(z)| \frac{d\theta}{z-x}$$

Therefore, $|h_R(x)| \leqslant \frac{1}{2\pi} \cdot \frac{2\varrho}{R-\varrho} \int^{2\pi} |\log|g_R(z)|| d\theta$

Now,

$$\log|g_R(z)|=\log|f(z)|-\log|f(o)|+\sum_{\nu=1}^{N} \log\left|1-\frac{z}{b_\nu}\right| - \sum_{\nu=1}^{M} \log\left|1-\frac{z}{a_\nu}\right|$$

* E. C. Titchmarsh: The Theory of Functions. p. 125.

And we have,

$$\left|1-\frac{z}{a_\nu}\right| \leqslant \frac{2R}{|a_\nu|}, \quad \left|1-\frac{z}{b_\nu}\right| \leqslant \frac{2R}{|b_\nu|}$$

Hence,

$$\left| \sum_{|b_\nu|<R} \log\left|1-\frac{z}{b_\nu}\right| - \sum_{|a_\nu|<R} \log\left|1-\frac{z}{a_\nu}\right| \right|$$

$$\leqslant \left| \Sigma \log \frac{2R}{|a_\nu|} \right| + \left| \Sigma \log \frac{2R}{|b_\nu|} \right|$$

$$\leqslant N(2R, o) + N(2R, \infty)$$

$$< 2T(2R) + O(1) \qquad (2\cdot1)$$

Again,

$$\frac{1}{2\pi} \int_o^{2\pi} |\log|f(z)|| \, d\theta = m(R, o) + m(R, \infty)$$

$$< 2T(R) + O(1)$$

$$< 2T(2R) + O(1) \qquad (2\cdot2)$$

Using (2·1) and (2·2), we get,

$$\frac{1}{2\pi} \int_o^{2\pi} |\log|g_R(z)|| \, d\theta \leqslant 4T(2R) + O(1).$$

Hence,

$$|h_R(x)| \leqslant [4T(2R) + O(1)] \frac{2\varrho}{R-\varrho}$$

We deduce therefore†

$$|h^1_R(x)| \leqslant [4T(2R) + O(1)] \frac{4(R+\varrho)}{(R-\varrho)^2}$$

which leads to the lemma.

3. Consider the integral

$$I = \int \frac{f^1(z)}{f(z)} \frac{dz}{z^q(z-x)}$$

taken along the circle $|z| = \frac{R}{2}$

Calculating the residues, we get,

$$\frac{1}{2\pi i} \int \frac{f^1(z)}{f(z)} \frac{dz}{z^q(z-x)} = \frac{f^1(x)}{f(x)} \frac{1}{x^q} + \sum_{|a_\nu| < R/2} \frac{1}{a_\nu^q (a_\nu - x)}$$

$$- \sum_{|b_\nu| < R/2} \frac{1}{b_\nu^q (b_\nu - x)} + \frac{1}{(q-1)!} \left(\frac{d^{q-1}}{dz^{q-1}} \right) \frac{f^1(z)}{f(z)(z-x)}_{z=0} \tag{3·1}$$

We may suppose, without loss of generality that $|z| = R/2$ is free from every a_ν and b_ν. Then,

$$\frac{1}{2\pi i} \int \frac{f^1(z)}{f(z)} \frac{dz}{z^q(z-x)}$$

† Ibid: p. 176.

$$=\frac{1}{2\pi i}\int_{|z|=R/2}\left(\frac{f^1(z)}{f(z)}-\sum_1^M\frac{1}{z-a_\nu}+\sum_1^N\frac{1}{z-b_\nu}\right)\frac{dx}{z^q(z-x)}$$

$$+\frac{1}{2\pi i}\sum_1^M\int_{|z|=R/2}\frac{1}{z-a_\nu}\frac{dz}{z^q(z-x)}$$

$$-\frac{1}{2\pi i}\sum_1^N\int_{|z|=R/2}\frac{1}{z-b_\nu}\frac{dz}{z^q(z-x)}$$

$$=I_1+I_2+I_3,\ \text{say.}$$

From the lemma proved above,

$$I_1=O\left(\frac{T(2R)}{R^{q+1}}\right)=O(1). \tag{3·2}$$

If k denotes the number of a_ν's lying inside $|z|=\frac{R}{2}$

$$I_2=\sum_{\nu=1}^{k}+\sum_{k+1}^{M}=-\sum_{R>|a_\nu|>R/2}\frac{1}{a_\nu^q(a_\nu-x)} \tag{3·3}$$

by contour integration.

Similarly,

$$I_3=\sum_{R>|b_\nu|>R/2}\frac{1}{b_\nu^q(b_\nu-x)} \tag{3·4}$$

Using (3·2), (3·3) and (3·4) we deduce from (3·1), that

$$\lim_{R=\infty}\left[\frac{f^1(x)}{f(x)}\frac{1}{x^q}+\sum_{|a_\nu|<R}\frac{1}{a_\nu^q(a_\nu-x)}-\sum_{|b_\nu|<R}\frac{1}{b_\nu^q(b_\nu-x)}\right.$$

$$\left.+\sum cx_n^{-n}\right]=o, \qquad (3\cdot5)$$

when R tends to infinity through such a sequence of values as will make $|z|=\frac{R}{2}$ free from any a_ν or b_ν.

That is,

$$\frac{f^1(x)}{f(x)}=\lim_{R=\infty}\left[\sum_{|a_\nu|<R}\left(\frac{x}{a_\nu}\right)^q\frac{1}{x-a_\nu}-\sum_{|b_\nu|<R}\left(\frac{x}{b_\nu}\right)^q\frac{1}{x-b_\nu}\right]$$

$$+\sum_{o}^{q-1}d_nx^n \qquad (3\cdot6)$$

where $d_n=-c_n$.

Now,

$$\left(\frac{x}{a_\nu}\right)^q\left(\frac{1}{x-a_\nu}\right)=\frac{x^{q-1}}{a^q}+\frac{x^{q-2}}{a_\nu^{q-1}}+\cdots\cdots$$

$$+\frac{1}{a_\nu}+\frac{1}{x-a_\nu}$$

Substituting this in (3·6), integrating, and raising each side to the power of e, we obtain the required result.

If $f(o)=o$ or ∞, we write $f(x)=x^a\psi(x)$, where a is a positive or negative integer, and apply a similar method to $\psi(x)$ in order to get the required result.

Remark. In (3·6) we see that the expression

$$\lim_{R=\infty}\left[\sum_{|a_\nu|<R}\frac{1}{a_\nu^q(a_\nu-x)}-\sum_{|b_\nu|<R}\frac{1}{b_\nu^q(b_\nu-x)}\right]$$

can be replaced by

$$\sum_1^\infty\frac{1}{a_\nu^q(a_\nu-x)}-\sum_1^\infty\frac{1}{b_\nu^q(b_\nu-x)}$$

provided $\sum\frac{1}{|a_\nu|^{q+1}}$ and $\sum\frac{1}{|b_\nu|^{q+1}}$

are convergent; which is equivalent to the statement that

$\int_0^\infty\frac{T(r)}{r^{q+2}}dr<\infty$, in which case,

$$f(x)=e^{\sum_0^q p_\nu x^\nu}\frac{\prod_1^\infty\left(1-\frac{x}{a_\nu}\right)e^{\frac{x}{a_\nu}+\cdots\cdots\frac{x^q}{qa_\nu^q}}}{\prod_1^\infty\left(1-\frac{x}{b_\nu}\right)e^{\frac{x}{b_\nu}+\cdots\cdots+\frac{x^q}{qb_\nu^q}}}$$

A NOTE ON GRAPHS:

The Presentation and Analysis of Changes in Proportion, Rates of Change and Changes in Rates of Change.

BY

R. GALLETTI, I.C.S.

As an introduction to this note I refer the reader to the remarks of Marshall in Money, Credit and Commerce, III. ii.6. The whole discussion, though concerned principally with the statistics of trade, should be read and the conclusion appreciated:

"Aggregates and percentages must then be studied together and not separately. Practice will enable anyone, while reading down columns of aggregate values for successive years, to interpolate as he goes fairly accurate estimates of percentage changes and *vice versa*".

As usual in Marshall, a telling remark is relegated to a footnote:

"This class of consideration (i.e., that increases of 10 per cent. in the commercial class of Russia and England would signify very different changes) suggests that the use of logarithmic diagrams, and similar devices for concentrating attention on changes in percentages of economic quantities, have but narrow uses".

Marshall in this place is discussing the data of commerce. But the same considerations hold good in regard to a very large part of the statistical field—in general, all that part in which changes,

*See also Marshall, The Graphic Method of Statistics (reprinted in Memorials of Alfred Marshall, pp. 175 ff)—E. C. Rhodes—Elementary Statistical Methods—Chapters 11 and 12—Allen—Mathematical Analysis for Economists—pp. 222 ff.

relative magnitudes, and most particularly changes in relative magnitudies, require attention. My own interest in this matter started with the analysis and presentation of figures of acreage and production for the study of agricultural land utilization. But it can be seen that in studies of industrial and commercial development, of banking deposits and credits, of changes in population and demography, of price, cost and production indices—not to mention a great many more—the relative changes of components with respect to each other and to the total have to be considered and presented; and if the total itself is changing there is no good way, if we use the methods of graphic statistics at present commonly used, of showing both absolute and relative changes in one figure. It is true that the changes can be shown in separate figures and studied side by side, like the columns of a table. But with all due deference to Marshall, even long practice does not make it particularly easy for ordinary students to see absolutes in terms of percentages or percentages in terms of absolutes.

I therefore suggest the use of two types of chart which are extremely simple to prepare, mathematically without complication, very useful in the analysis of data of change (especially when rough results will do and accuracy to several places of decimals is otiose) because they save calculation and emphasize relationships, and a complete solution of the problem of representing the absolute and the percentage figure, whether of a component or of a change or of a change in a component, by the same visual feature.

Both types are based on the properties of similar triangles and on the principle of showing the relationship of the variables to each other and not their relation to time intervals. The ordinary type of chart, whether arithmetic or logarithmic, and whether showing the variables by lines, points, or areas, uses one axis for time, so that it cannot represent the functional relationship between variables in a two-dimensional figure. The types I advocate relegate time to the status of a condition of change, and take care of the intervals in the selection of the data (which, as is usual in time-series, are given for equal intervals—year, month, or whatever

suits the particular study) and of the order by connecting points in sequence. For certain purposes the time-intervals need not be equal. Successive changes can be studied without studying the changing rates of change.

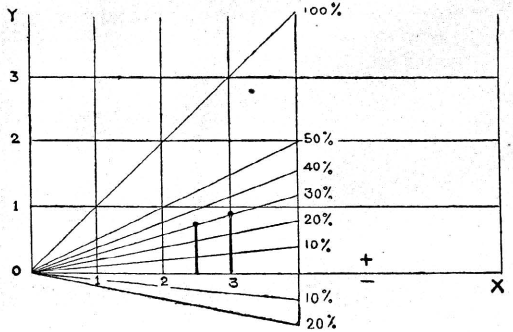

Fig. I.

The first type is shown in Fig. I. The ordinates being drawn perpendicular to the x axis, i.e., parallel to each other, the triangles formed by the ordinates joining the x axis to any single radius vector are similar, and the proportion of each ordinate to each abscissa is the same. If (as I think simplest and best) the scale of the two axes is the same, the proportion borne by the ordinate to the abscissa becomes immediately apparent. The 45° radius vector is the locus of all ordinates which are equal to 100% of the abscissa; and if any such ordinate is divided into ten equal parts, the radius vectors passing through the points of division will be the loci of 10% intervals. The absolute scale is read as usual on the y axis; the percentages are read on the radius vectors, which can easily be drawn for 10% or even 5% intervals without unduly confusing the chart.

When changes are in question and are sometimes in one direction and sometimes in the other it is necessary to use two quadrants, marking one + and the other − .

For deviations from a fixed base, i.e., when there is only one variable to consider, this chart is not so good as one in which time

occupies one axis; but for the charting of link relatives and first differences (which is more common in handling time series, at any rate in the groups of studies mentioned above) they yield graphs visually, logically and mathematically superior, since they put variations in base and in deviation into a mathematically functional relation (easily expressed in an equation,) the general shape of the graph of which reveals the nature of the trend of deviation, clearly distinguishing between e.g., steady trends of change (which are linear along a radius vector), accelerating or decelerating trends of change (which are curves of easily recognizable pattern), alternating fluctuations and their various types (divergent and convergent), and cyclic or spiral movements with their phases or increasing—decreasing—increasing rates of changes. The rate of change at each interval is immediately read with the aid of the percentage scale, and changes in rates of change are manifest from the deviation of the course of the graph from the radius vectors.

In the appendices are given some examples of the use of this type of graph. Such charts may help statisticians to avoid wasting time in fitting mathematical curves to series which do not really admit of them. I hope the mathematicians will be interested enough to work on these curves, which, though not continuous functions, are capable, I believe, of indicating the continuous functions when they exist.

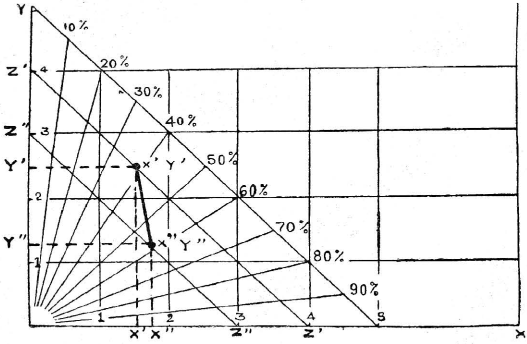

Fig. II.

The other type of chart advocated is shown in Fig. 2. Here the two components of a total are plotted as ordinate and abscissa; if there are more than two components, the abscissa can represent the one particularly studied and the ordinate then represents the sum of the rest. If the total is z, the point (x, y) must, whatever the value of x and the value of y, fall on a straight line joining the point z on the x axis with the point z on the y axis; and if the scale is the same on both axes, the locus of all possible $(x+y)s$ is a line at 45° to the axes. All possible totals are represented by an infinite number of such lines, all parallel to each other and therefore forming with the axes a series of similar triangles. Any radius vector divides every such line in the same proportion. By dividing any such line into ten equal portions we provide a scale of 10% intervals. The radius vectors through the points of division cross every 'total locus' at the same 10% intervals, so that in charting the absolute value of any x against its appropriate y we also chart it as a percentage of the appropriate z. Every other component of the same z can be charted in the same way, so that the percentage as well as the absolute value of every component is shown very clearly on one chart. As z changes through time, the changes of x and y are both shown. Z can be not only a changing total in a time series but various totals of the same kind at one period—e.g., the foreign trade of different countries in a particular year, or the foreign trade of various countries in a particular commodity. In certain instances a very large number of components can be charted on the same chart without confusion; and since groups of components are necessarily smaller than the total of all components, sections of the whole total can often be conveniently analysed along with the aggregate.

This chart is visually very effective once its logic and mathematics are understood. It draws attention to substitution effects (shown by the slope of the line joining two points in a time series diagram) and enables us to measure by the relative change in the angle of the radius vectors the real degree of change. For instance, it is usual to express a change in the acreage of a particular crop (1) as a percentage of its own base figure; (2) by the percentage the acreage was of an earlier total; (3) by the comparison of the change in the crop with the change in the total. To present these facts graphically by the ordinary methods at least three diagrams are required. If rates of change over more than one interval are to be shown, an arithmetical chart for absolute change and a

logarithmic chart for rates of change are added. With charts of the type here advocated, two charts, one of each type, will present all the relationships, offer a measure of relative change—e.g.,

$$\frac{x'}{y'} \Big/ \frac{x''}{y''} \text{ or } \frac{x'}{z'} \Big/ \frac{x''}{z''} \text{ or } \frac{\tan \theta_0}{\tan \theta_1}$$

and throw in the approximate elasticity of substitution.

Mathematically, the graph presents a function of two variables (for time series—not where the chart is used to present compendiously relative magnitudes in a given situation) and indication by its form the law (or want of law) of change in one component relatively to others. In this second type of chart the function indicated by the curve may be continuous and intorpolation is often permissible. Some examples of the use of this kind of graph are given in the appendix.

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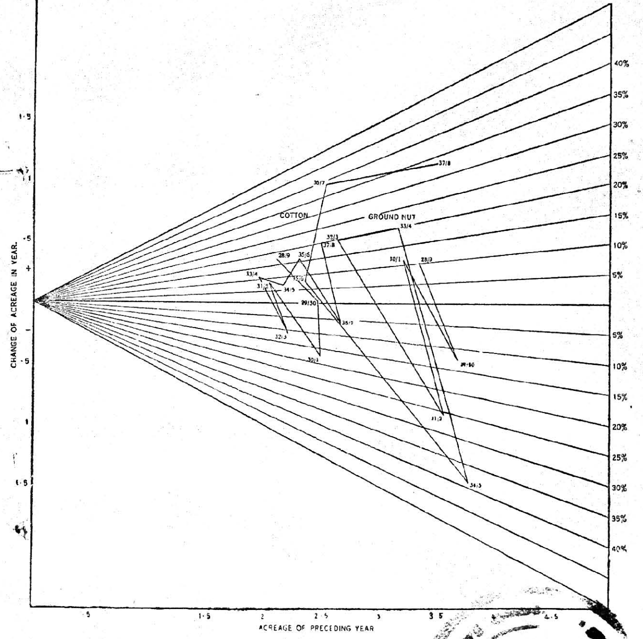
MADRAS PRESIDENCY: — CHANGES IN ACREAGE OF GROUND NUT & COTTON 1927/8 TO 1937/8



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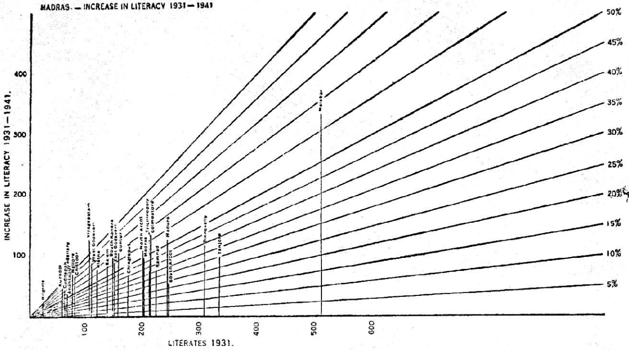
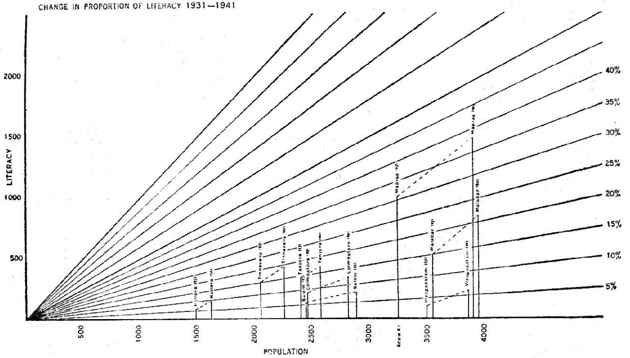
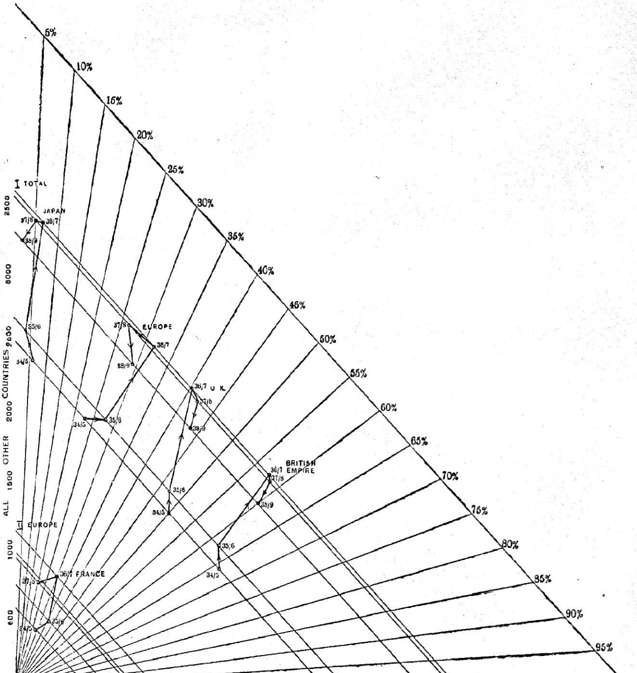
MADRAS. — INCREASE IN LITERACY 1931—1941

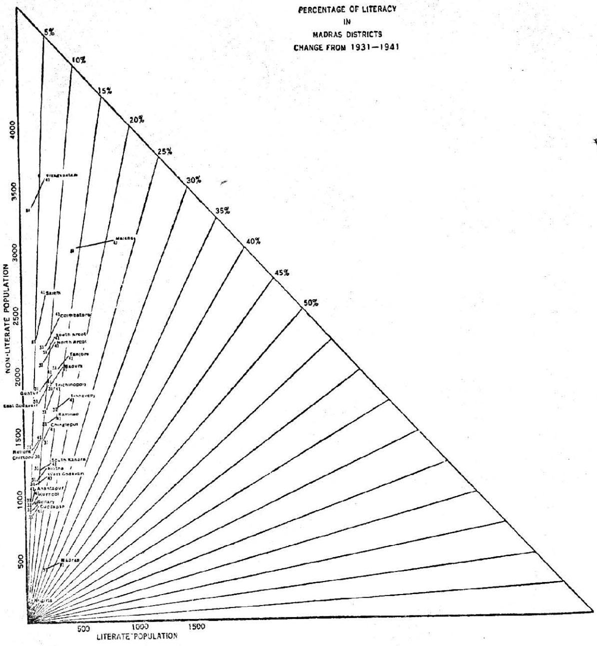


CHANGE IN PROPORTION OF LITERACY 1931—1941



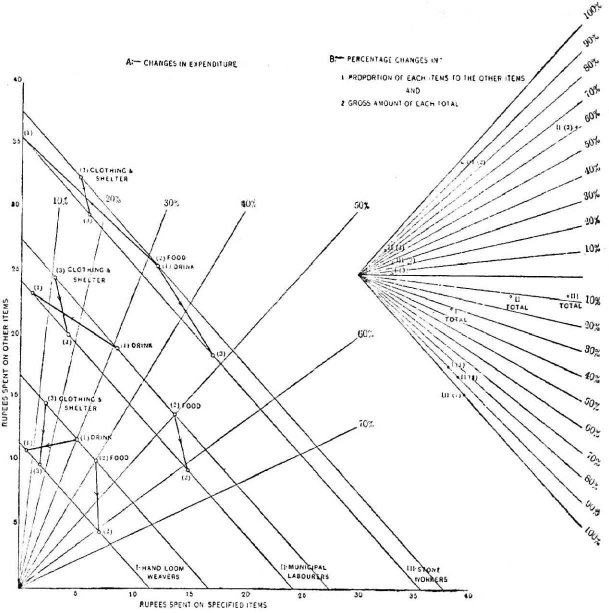
QUINQUENNIAL STATISTICAL ABSTRACT FOR THE MADRAS PRESIDENCY
MADRAS FOREIGN TRADE EXPORTS (BURMA EXCLUDED)




PERCENTAGE OF LITERACY
IN
MADRAS DISTRICTS
CHANGE FROM 1931—1941
5%
10%
15%
20%
25%
30%
35%
40%
45%
50%
NON-LITERATE POPULATION
4000
3500
3000
2500
2000
1500
1000
500
LITERATE POPULATION
500
1000
1500
Salem
Coimbatore
South Arcot
North Arcot
Tanjore
Madura
Trichinopoly
Guntur
East Godavari
Chingleput
Nellore
Chittoor
South Kanara
West Godavari
Anantapur
Kurnool
Cuddapah
Madras

(SANKHYA IV I P 456)

SHIFTS IN EXPENDITURE AFTER PROHIBITION IN SALEM



ON THE INTERVAL BETWEEN THE RANKED INDIVIDUALS OF SAMPLES TAKEN FROM A RECTANGULAR POPULATION*

By

D. V. RAJALAKSHMAN,
University of Madras.

Introduction:

J. Neyman and E. S. Pearson[1] in establishing the criteria for testing the hypothesis for a sample that has been drawn at random from a specified population, obtained[2] the chance of drawing a sample with range in the limits $W \pm \frac{dW}{2}$ from a rectangular population, where W is the range $(x_n - x_1)$ of a sample of size n, the characters of the n observations of the sample being $x_1, x_2 \ldots . x_n$, such that

$$x_1 \leqslant x_2 \leqslant \ldots \ldots . \quad \leqslant x_n$$

and measured from one end of the range w of the infinite population. Also Karl Pearson(2) derived an expression for the sth moment of the interval x between q^{th} *and* q^{1th} ranked individuals to be

$$\mu^1_s = \omega^s \frac{n!\,(q^1 - q - 1 + s)!}{(n+s)!\,(q^1 - q - 1)!} \qquad \ldots \quad (1)$$

*Taken from a thesis accepted for the M.Sc. degree of the University of Madras.

1. Numbers in [ ] indicates references at the end.

2. [1] p. 210.

about one end of the range w of the rectangular population. By taking the first four moments in particular, he obtained the distribution of the interval x to be

$$y = \frac{N}{\omega} \cdot \frac{n!}{(q^1-q-1)!(n-q^1+q)!} \left(\frac{q^1-q-1}{n-1} + \frac{x}{\omega} \right)^{q^1-q-1} \left(\frac{n-q^1+q}{n-1} - \frac{x}{\omega} \right)^{n-q^1+q} \quad .. \quad (2)$$

where the origin is the mode of the distribution when N samples of size n are taken, from the specified population. In this paper the distribution of the interval between the qth and q^{1th} ranked individuals is obtained directly by adopting a method similar to that of Neyman and Pearson, and also the mth moment of this interval[3] about the arithmetic mean of the intervals of samples is obtained and some special features of this distribution are discussed.

Distribution of the Interval:

Assuming the original population to be infinite and all the values of the variable between o and w to be of equal frequency of occurrence in the population, the chance of obtaining one observation at $x_{q^1} \pm \frac{dx_{q^1}}{2}$ one at $x_q \pm \frac{dx_q}{2}$, $(q-1)$ observations below x_q, (q^1-q-1) observations between x_{q^1} and x_q and $(n-q^1)$ observations above x_{q^1} is

$$p = \frac{n!}{(q-1)!\,(q^1-q-1)!\,(n-q^1)!} \left(\frac{dx_{q^1}}{\omega} \right) \left(\frac{dx_q}{\omega} \right) \left(\frac{x_q}{\omega} \right)^{q-1} \left(\frac{x_{q^1}-x_q}{\omega} \right)^{q^1-q-1} \left(\frac{\omega - x_{q^1}}{\omega} \right)^{n-q^1}$$

where $q^1 > q$

3. 'Interval' is used for the range between the q^{1th} and q^{th} ranked individuals throughout this paper.

$$\text{or } p = \frac{1}{\omega^n} \cdot \frac{n!}{(q-1)!\,(q^1-q-1)!\,(n-q^1)!}\, x_q^{q-1}\, (x_{q^1}-x_q)^{q1-q-1}$$

$$(\omega - x_{q^1})^{n-q^1} \cdot dx_q \cdot dx_{q^1} \qquad .. \quad (3)$$

changing the variables to $G^4 = \frac{1}{2}(x_{q^1}+x_q)$ and $W = (x_{q^1}-x_q)$

so that we have $dG \cdot dW = dx_{q^1} \cdot dx_q$

Then the chance of having a sample with centre and interval

in the limits $G \pm \frac{dG}{2}$ and $W \pm \frac{dW}{2}$ is

$$\varphi_1(W, G)\, dW \cdot dG = \frac{k}{\omega^n} \cdot W^{q^1-q-1} \left(\frac{2G-W}{2}\right)^{q-1}$$

$$\left(\omega - \frac{2G+W}{2}\right)^{n-q^1} dW \cdot dG$$

where $k = \dfrac{n!}{(q-1)!\,(q^1-q-1)!\,(n-q^1)!}$

When W is fixed G can vary between the limits $\frac{W}{2}$ and $\omega - \frac{W}{2}$ so that the chance $\varphi_2(W)\, dW$ of drawing a sample with interval in the limits $W \pm \frac{dW}{2}$ is obtained by integrating for G in the limits specified.

4. G is termed as the centre of the interval between the q^{1th} and q^{th} ranked individuals.

So, $$\varphi_2(W) = \int_{W/2}^{(\omega - W/2)} \frac{k}{\omega^n} \cdot W^{q^1-q-1} \left(\frac{2G-W}{2} \right)^{q-1} \left(\omega - \frac{2G+W}{2} \right)^{n-q^1} dG.$$

Using the transformation $G = \frac{W}{2} + (\omega - W)z$, the integral reduces to

$$\varphi_2(W) = \frac{k}{\omega^n} \cdot W^{q^1-q-1} \int_0^1 (\omega - W)^{q-1} \cdot z^{q-1} \cdot (\omega - W)^{n-q^1} \cdot (1-z)^{n-q^1} (\omega - W)\, dz$$

$$= \frac{k}{\omega^n} \cdot W^{q^1-q-1} \cdot (\omega - W)^{n-q^1+q} \cdot B(q, n-q^1+1).$$

As n, q^1 and q are positive integers, we get

$$\varphi_2(W) = \frac{n!}{(q^1-q-1)!\,(n-q^1+q)!} \cdot \frac{1}{\omega} \cdot \left(\frac{W}{\omega} \right)^{q^1-q-1} \left(1 - \frac{W}{\omega} \right)^{n-q^1+q}$$

So, the distribution of W, the interval, when N samples of size n are taken from the rectangular population will be

$$y = N \cdot \varphi_2(W)$$

or, writing x for W, the equation for the distribution of the interval will be

$$y = \frac{N}{\omega} \cdot \frac{n!}{(q^1-q-1)!\,(n-q^1+q)!} \cdot \left(\frac{x}{\omega} \right)^{q^1-q-1} \left(1 - \frac{x}{\omega} \right)^{n-q^1+q} \quad \ldots (4)$$

where the origin is the one end of the range w of the infinite population under consideration. Shifting the origin to $\left(\frac{q^1-q-1}{n-1}\,\omega,\ 0\right)$ the mode, we can easily verfiy that this equation reduces itself to (2) derived by Karl Pearson.

m^{th} moment of the interval about the mean interval using the distribution function:

Having $s=1$ in equation (1) the mean interval of the samples reduces to

$$\bar{x}_{(q^1.\,q)} = \frac{(q^1-q)}{n+1}\cdot\omega \qquad \text{.. (5)}$$

If μ_m be the mth moment of the interval about the mean $\bar{x}_{(q^1.\,q)}$ where m is a positive integer, then

$$\mu_m \int_0^\omega y\,dx = \int_0^\omega y\,(x-\bar{x}_{(q^1q)})^m dx$$

$$\text{or } \mu_m \int_0^\omega \left(\frac{x}{\omega}\right)^{q^1-q-1}\left(1-\frac{x}{\omega}\right)^{n-q^1+q} dx = \int_0^\omega \left(\frac{x}{\omega}\right)^{q^1-q-1}$$

$$\left(1-\frac{x}{\omega}\right)^{n-q^1+q}\left(x-\frac{q^1-q}{n+1}\,\omega\right)^m dx$$

Putting $x=\omega z$, this reduces to

$$\mu_m\cdot B(q^1-q,\ n-q^1+q+1) = (-\omega)^m \int_0^1 z^{q^1-q-1}\,(1-z)^{n-q^1+q}$$

$$\left(\frac{q^1-q}{n+1}-z\right)^m dz \qquad \text{.. (6)}$$

expanding the last term on the right hand side, we have

$$=(q^1-q)^m\left(-\frac{\omega}{n+1}\right)^m \int_0^1 z^{q^1-q-1}\,(1-z)^{n-q^1+q}$$

$$\left[1-\frac{m(n+1)}{(q^1-q)}z+\frac{m(m-1)}{1\cdot 2}\,\frac{(n+1)^2}{(q^1-q)^2}z^2-\ldots\right] dz$$

$$=(q^1-q)^m\left(-\frac{\omega}{n+1}\right)^m\left[B(q^1-q,\, n-q^1+q+1)\right.$$

$$\left.-\frac{m\,(n+1)}{(q^1-q)}B(q^1-q+1,\, n-q^1+q+1)+\ldots\ldots\right]$$

$$=(q^1-q)^m\left(-\frac{\omega}{n+1}\right)^m\frac{(q^1-q-1)!\,(n-q^1+q)!}{n!}$$

$$\left[1-\frac{m\,(q^1-q)}{(n+1)}\times\frac{(n+1)}{(q^1-q)}\right.$$

$$\left.+\frac{m\,(m-1)}{1\cdot 2}\,\frac{(q^1-q)\,(q^1-q+1)}{(n+1)\,(n+2)}\left(\frac{n+1}{q^1-q}\right)^2-\ldots\ldots\ldots\right]$$

Hence we have

$$\mu_m=(q^1-q)^m\left(-\frac{\omega}{n+1}\right)^m$$

$$F\left[-m,\ (q^1-q);\ (n+1);\ \frac{(n+1)}{(q^1-q)}\right] \quad \ldots \quad (7)$$

Where F $(a, b; c; z)$ is the standard hypergeometric series

$$1+\frac{a\cdot b}{1\cdot c}z+\frac{a\,(a+1)\cdot b\,(b+1)}{1\cdot 2\cdot c\,(c+1)}z^2+\ldots\ldots\ldots\ldots\ldots\ldots$$

Also in (6) writing the factor

$$\left(z-\frac{q^1-q}{n+1}\right)^m \text{ as } \left[\frac{(n-q^1+q+1)}{n+1}-(1-z)\right]^m$$

and then continuing the integration after expanding this factor, we get the result

$$\mu_m=(n-q^1+q+1)^m\left(\frac{\omega}{n+1}\right)^m$$

$$\times F\left[-m,\ (n-q^1+q+1);\ (n+1);\ \frac{(n+1)}{(n-q^1+q+1)}\right] \quad \ldots \quad (8)$$

Where F $(a, b; c; z)$ is the usual hypergeometric series. From the elementary property[5] of the hypergeometric function the results (7) and (8) can be noted be the same.

m^{th} moment of the interval from probability considerations.

The expression for μ_m can also be derived directly from the the chance p obtained in equation (3). The value of x_q ranges between o to x_{q^1} and x_q can be anywhere between o to ω.

Hence $\bar{x}_{(q^1q)}$, the mean interval of the samples, becomes.

$$\bar{x}_{(q^1q)} = \frac{c}{\omega^n} \int_o^{\omega} (\omega - x_{q^1})^{n-q^1} dx_{q^1} \int_o^{x_{q^1}} x_q^{\,q-1} (x_{q^1} - x_q)^{q^1-q} dx_q$$

where $c = \dfrac{n!}{(q-1)!\,(q^1-q-1)!\,(n-q^1)!}$

Let $I = \int_o^{x_{q^1}} x_q^{\,q-1} (x_{q^1} - x_q)^{q^1-q} dx_q$

Put $x_q = x_{q^1} \cdot z$.

Then $I = \int_o^1 x_{q^1}^{q^1} \cdot z^{q-1} (1-z)^{q^1-q} dz = x_{q^1}^{q} B(q, q^1-q+1).$

So, $\bar{x}_{(q^1\,q)} = \dfrac{c}{\omega^n} B(q, q^1-q+1) \int_o^{\omega} x_{q^1}^{q} (\omega - x_{q^1})^{n-q^1} dx_{q^1}$

---

5. The property referred to is

$$F(A, B; C; x) = \left(\frac{1}{1-x}\right)^A F\left(A, C-B; C; \frac{x}{x-1}\right)$$

Substituting $x_{q^1}=\omega z$, and integrating we have

$$\bar{x}_{(q^1q)}=\frac{(q^1-q)}{(n+1)}\omega \qquad \ldots (10)$$

which is the same as (5).

Also μ_m, the mth moment of the interval about the mean $\bar{x}_{(q^1\,q)}$ becomes

$$\mu_m=\frac{c}{\omega^n}\int_o^{\omega} dx_{q^1}(\omega-x_{q^1})^{n-q^1}\int_o^{x_{q^1}} x_q^{q-1}(x_{q^1}-x_q)^{q^1-q-1}$$

$$\times\left(x_{q^1}-x_q-\frac{q^1-q}{n+1}\omega\right)^m dx_q \qquad \ldots (11)$$

Let $I=\int_o^{x_{q^1}} x_q^{q-1}\,(x_{q^1}-x_q)^{q^1-q-1}\left(x_{q^1}-x_q-\frac{q^1-q}{n+1}\omega\right)^m dx_q$

Put $x_q=x_{q^1}\cdot z$, then

$$I=(-1)^m\int_o^1 x_{q^1}^{q^1-1}\cdot z^{q-1}(1-z)^{q^1-q-1}\left[\frac{q^1-q}{n+1}\omega-x_{q^1}(1-z)\right]^m dz$$

$$=(-1)^m x_{q^1}^{q^1-1}\int_o^1 z^{q-1}(1-z)^{q^1-q-1}\left[\left(\frac{q^1-q}{n+1}\omega\right)^m\right.$$

$$-m\left(\frac{q^1-q}{n+1}\omega\right)^{m-1}x_{q^1}(1-z)+\frac{m(m-1)}{1\cdot 2}\left(\frac{q^1-q}{n+1}\omega\right)^{m-2}$$

$$\left.\times\, x_{q^1}^2(1-z)^2-\ldots\right]\ldots dz$$

Substituting the values of B-function, we have

$$I=\omega^m(-1)^m\left(\frac{q^1-q}{n+1}\right)^m\frac{(q-1)!\,(q^1-q-1)!}{(q^1-1)!}x_{q^1}^{q^1-1}$$

$$\left[1-m\left(\frac{n+1}{q^1-q}\,\omega\right)x_{q^1}\left(\frac{q^1-q}{q}\right)\right.$$

$$\left.+\frac{m(m-1)}{1\cdot 2}\left(\frac{n+1}{q^1-q}\,\omega\right)^2\times x_{q^1}^2\cdot\frac{(q^1-q)(q^1-q+1)}{q^1(q^1+1)}-\ldots\right]$$

Put $c^1=(-1)^m\left(\dfrac{q^1-q}{n+1}\right)^m\dfrac{(q-1)!\,(q^1-q-1)!}{(q^1-1)!}$

Then $\mu_m=\dfrac{c}{\omega^n}\int_0^\omega(\omega-x_{q^1})^{n-q^1}\cdot I\cdot dx_{q^1}$

using the substitution $x=\omega z$, this reduces to

$$\mu_m=\frac{\omega^m}{\omega^n}cc^1\int_0^1\omega^n(1-z)^{n-q^1}z^{q^1-1}\left[1-m\left(\frac{n+1}{q^1-q}\right)\left(\frac{q^1-q}{q}\right)z\right.$$

$$\left.+\frac{m(m-1)}{1\cdot 2}\left(\frac{n+1}{q^1-q}\right)^2\frac{(q^1-q)(q^1-q+1)}{q^1(q^1+1)}z^2\ldots\right]dz$$

$$=\omega^m cc^1\left[\frac{(n-q^1)!\,(q^1-1)!}{n!}-m\left(\frac{n+1}{q^1-q}\right)\frac{(q^1-q)}{q^1}\frac{(n-q^1)!\,q^1!}{(n+1)!}\right.$$

$$+\frac{m(m-1)}{1\cdot 2}\left(\frac{n+1}{q^1-q}\right)^2\cdot\frac{(q^1-q)(q^1-q+1)}{q^1(q^1+1)}$$

$$\left.\times\frac{(n-q^1)!\,(q^1+1)!}{(n+2)!}-\ldots\ldots\right]$$

$$=\omega^m \cdot cc^1 \cdot c^{11}\left[1-m\left(\frac{n+1}{q^1-q}\right)\left(\frac{q^1-q}{n+1}\right)\right.$$

$$\left.+\frac{m(m-1)}{1\cdot 2}\left(\frac{n+1}{q^1-q}\right)^2 \frac{(q^1-q)(q^1-q+1)}{(n+1)(n+2)}-\ldots\ldots\right]$$

where $c^{11}=\dfrac{(n-q^1)!\,(q^1-1)!}{n!}$

The constant $c \cdot c^1 \cdot c^{11}$ reduces to $(q^1-q)^m\left(-\dfrac{\omega}{n+1}\right)^m$

so that

$$\mu_m=(q^1-q)^m\left(-\frac{\omega}{n+1}\right)^m F\left[-m,\ (q^1-q);\ (n+1);\ \frac{n+1}{q^1-q}\right] \quad \ldots (12)$$

where F $(a,\ b;\ c;\ z)$ is the standard hypergeometric series. This is the same as the expression obtained in (7). Also using the substitution $x_q=(1-z)\ x_{q1}$ in (11) and continuing the integration the value for μ_m can be derived to be the same as in (8).

So we have the two forms of the mth moments of the interval between the q^{1th} and q^{th} ranked individuals about the mean interval for samples of size n taken from a rectangular distribution defined by the range w, to be

$$\mu_m^{(q^1q)}=(-1)^m(q^1-q)^m\left(\frac{\omega}{n+1}\right)^m$$

$$F\left[-m,\ (q^1-q);\ n+1;\ \frac{n+1}{q^1-q}\right] \text{———} A_{(q^1q)}$$

$$=(n-q^1+q+1)^m\left(\frac{\omega}{n+1}\right)^m$$

$$F\left[-m,\ (n-q^1+q+1);\ (n+1);\ \frac{n+1}{n-q^1+q+1}\right] \text{——} B_{(q^1q)}$$

Particular Cases:

(*i*) The two forms of the mth moment of the individual x_q in the qth rank taken about the mean value of that individual may be obtained by having $q^1=q$ and $q=o$ in $A_{(q^1q)}$ and $B_{(q^1q)}$ to be

$$\mu_m^{(q)}=(-1)^m q^m \left(\frac{\omega}{n+1}\right)^m F\left[-m, q; n+1; \frac{n+1}{q}\right] \text{———} A_{(q)}$$

$$=(n-q+1)^m \left(\frac{\omega}{n+1}\right)^m F\left[-m, (n-q+1); (n+1); \frac{n+1}{n-q+1}\right] \text{———} B_{(q)}$$

These expressions can also be derived independently either by taking the distributive function for the qth ranked individual x_q, obtained by Pearson seperately[6] and also as a special case of biquadratic curves,[7] and performing the integration as for the interval, or by considering the chance, under usual assumptions, of having an observation at $x_q \pm \frac{dx_q}{2}$, $(q-1)$ observations below x_q and $(n-q)$ above x_q to be

$$p^1=\frac{n!}{(q-1)!\,(n-q)!}\,\frac{1}{\omega^n}\,(\omega-x_q)^{n-q}\;x_q^{q-1}dx_q$$

and integrating as usual by taking x_q to lie anywhere between o to ω.

(*ii*). The functions[8] for the mth moments of the characters of the individuals x_1 and x_n in the extreme ranks about their res-

6. [2] p. 391.
7. [3] p. 239.
8. These functions are derived independently in [4].

pective means can be had by putting $q=1$ in $A_{(q)}$ and $q=n$ in $B_{(q)}$ to be

$$\mu_m^{(1)}=(-1)^m\left(\frac{\omega}{n+1}\right)^m \quad F[-m,\ 1;\ n+1;\ n+1] \text{———} A_{(1)}$$

$$\mu_m^{(n)}=\left(\frac{\omega}{n+1}\right)^m \quad F[-m, 1; n+1; n+1] \quad \text{———} B_{(n)}$$

(*iii*). The mth moment of the distribution of the range (x_n-x_1) of samples about the mean range[9] is obtained by making the substitution $q^1=n$ and $q=1$ in $B_{(q)}$

$$\mu_m^{(n,\,1)}=\left(\frac{2\omega}{n+1}\right)^m F\left[-m, 2; n+1; \frac{n+1}{2}\right] \quad \text{———} B_{(n,\,1)}$$

(*iv*). We have also the mth moments of the distributions of the ranked individuals x_q and x_{n-q+1} about their respective means by putting $q=q$ in $A_{(q)}$ and $q=(n-q+1)$ in $B_{(q)}$ to be

$$\mu_m^{(q)}=(-1)^m q^m\left(\frac{\omega}{n+1}\right)^m F\left[-m, q; n+1; \frac{n+1}{q}\right] \text{——} A_{(q)}$$

$$\mu_m^{(n-q+1)}=q^m\left(\frac{\omega}{n+1}\right)^m F\left[-m, q; n+1; \frac{n+1}{q}\right] \text{——} B_{(n-q+1)}$$

The two functions above are the same except for the factor $(-1)^m$. This indicates that in general the mth moments of the characters of ranked individuals about their respective means, which are at equal ranking counted from either end differ only by the factor $(-1)^m$. When n is even, say $2p$, all the ranked individuals of the sample can be formed into p pairs, starting from the individuals at the either end having the mth moments of the distributions of individuals in each pair being the same except for a factor $(-1)^m$. When n is odd, say $(2\,k+1)$, all the ranked individuals can be formed into k pairs leaving out the median individual x_{k+1} taking from either end such that the mth moments of the distributions of the individuals in each pair differ only by the

9. This has been obtained seperately in [5].

factor $(-1)^m$. The mth moment of the median ranked individual about the mean-median of samples reduces to

$$\mu_m^{(k+1)} = \left(\frac{\omega}{2}\right)^m F\left[-m, k+1;\ n+1; \frac{n+1}{k+1}\right]$$

Where m is any even integer, all the odd moments being equal to zero.

All these cases can be explained from the curve indicating the distribution of the interval x between ranked individuals already given in equations (2) and (4). Giving particular values to q^1 and q it can be seen easily that the curves indicating the distributions of the pairs of ranked individuals formed by pairing individauls from either end in $x_1\ x_2 \ldots\ldots x_n$, are the reflections of each other in the interval o to w, the range of the original rectangular population. The curves for the extreme values are of Pearson's Type IX, and for the remaining pairs Type I, when n is even, and when n is odd the curve showing the distribution of the median rank x_{k+1}, is of Type II being symmetrical about half the original range. In this particular case the curve reduces to

$$y = \frac{N}{\omega}\ \frac{(2k+1)!}{k!\,k!}\left(\frac{1}{4} - \frac{x^2}{\omega^2}\right)^k$$

indicating that all the odd moments vanish.

(v). The mth moment of the interval between successive individuals can be had by putting $q=q$ and $q^1=q+1$ in $A_{(q1\ q)}$ to be

$$\mu_m^{(q+1\ q)} = (-1)^m \left(\frac{\omega}{n+1}\right)^m F[-m, 1;\ n+1;\ n+1] \text{ ——} A_{(q+1\ q)}$$

which is independent of q and is same as the mth moment of the distribution of the first individual x_1.

In general the mth moment of the interval between any fixed number of individuals can be had by putting $q=q$ and $q^1=q+k$ in $A_{(q1\ q)}$, where k is a positive integer, to be

$$\mu_m^{(q+k,\ q)} = (-1)^m k^m \left(\frac{\omega}{n+1}\right)^m F\left[-m, k;\ n+1; \frac{n+1}{k}\right]$$

which is independent of q and is same as the mth moment of the distribution of the kth ranked individual x_k. We note in particular that if n is odd, say $(4\ p+3)$, the mth moment of the distribution of the quartile interval is the same as the mth moment of the median which is the $(2\ p+2)^{th}$ ranked individual. Also the mth moment of the distribution of range (x_n-x_1) can be noted to be the same as the mth moment of the distribution of the $(n-1)^{th}$ ranked individual x_{n-1}. This can be seen from the distribution function for the interval given in (2), as it involves only the difference (q^1-q) between the ranks.

I am indebted to Dr. N. S. Sastry, Head of the Department of Statistics, University of Madras, for the suggestions offered during the preperation of this paper.

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